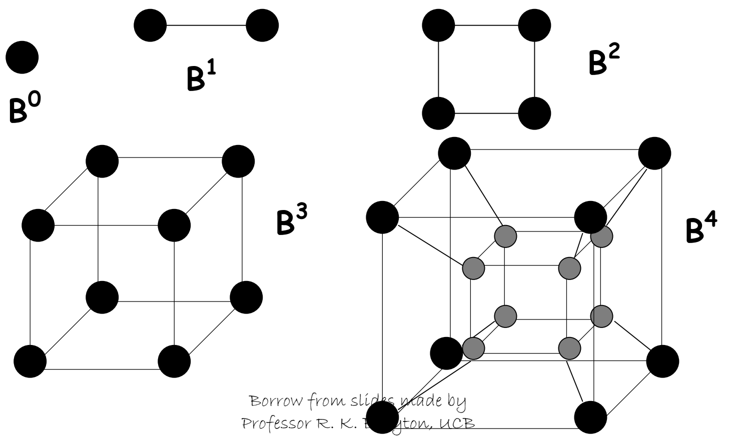
Logic Synthesis

Preliminary

The Boolean n-cube B^n

- $B = \{ 0,1\}$
- $B^2 = \{0,1\} \times \{0,1\} = \{00, 01, 10, 11\}$



Boolean Functions

$f(x) : B^n \longrightarrow B$

$B = \{0, 1\}$, $x = (x_1, x_2, \ldots, x_n)$

- $x_1, x_2, \ldots$ are variables
- $x_1, \overline{x_1}, x_2, \overline{x_2}, \ldots$ are literals
- each vertex of B^n is mapped to 0 or 1
- the onset of f is $\{x | f(x)=1\} = f^{1} = f^{-1}(1)$
- the offset of f is $\{x | f(x)=0\} = f^{0} = f^{-1}(0)$
- if $f^{1} = B^n$, f is the tautology, i.e. $f \equiv 1$
- if $f^{0} = B^n$ ($f^{1} = \emptyset$), f is not satisfiable
- if $f(x) = g(x)$ for all $x \in B^n$, then f and g are equivalent

We write simply f instead of f^{1}

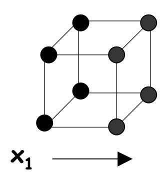
Literals

A literal is a variable or its negation

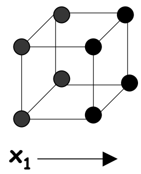
$y, \overline{y}$

It represents a logic function

Literal x_1 represents the logic function f, where
$f = \{x | x_1 = 1\}$



f = x_1



f = $\overline{x_1}$

Literal $\overline{x_1}$ represents logic function g where
$g = \{x | x_1 = 0\}$

Boolean Formulas

Boolean formulas can be represented by formulas defined as catenations of

- **parentheses (,)**
- **literals x, y, z, $\bar{x}$, $\bar{y}$, $\bar{z}$**
- **Boolean operators + (OR), X (AND)**
- **complementation, e.g. $\overline{x + y}$**

Examples

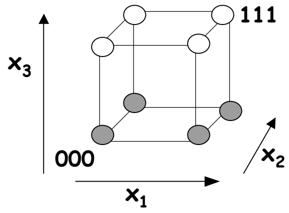
$$f = x_1 \times \bar{x}_2 + \bar{x}_1 \times x_2 = (x_1+x_2) \times (\bar{x}_1+\bar{x}_2)$$

$$h = a + b \times c = \bar{a} \times (\bar{b} + \bar{c})$$

We usually replace X by catenation, e.g. a X b→ab

Logic functions

- There are 2^n vertices in input space B^n



000		1	
001		0	
010		1	
011		0	"truth table"
100	⇒	1	
101		0	
110		1	
111		0	

- There are 2^{2^n} distinct logic functions.
  - Each subset of vertices is a distinct logic function: $f \subseteq B^n$

Logic Functions

- **However, there are infinite number of logic formulas**

$$\begin{aligned} f &= x + y \\ &= x\bar{y} + xy + \bar{x}y \\ &= x\bar{x} + x\bar{y} + y \\ &= (x + y)(x + \bar{y}) + \bar{x}y \end{aligned}$$

- **Synthesis = Find the best formula (or “representation”)**

Boolean Operations - AND, OR, COMPLEMENT

$$f : B^n \rightarrow B$$
$$g : B^n \rightarrow B$$

- AND - $fg = h$ such that
  $h = \{x \mid f(x)=1 \text{ and } g(x)=1\}$

- OR - $f + g = h$ such that
  $h = \{x \mid f(x)=1 \text{ or } g(x)=1\}$

- COMPLEMENT - $\bar{f} = h$ such that
  $h = \{x \mid f(x) = 0\}$

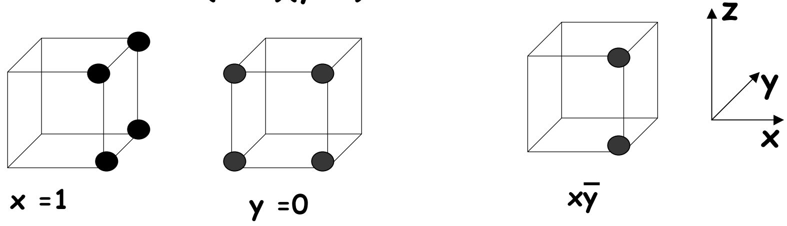
Cubes

- The AND of a set of literal functions ("conjunction" of literals) is a cube

$$C = x\overline{y}$$

is a cube

$$C = (x=1)(y=0)$$



Cubes

- If $C \subseteq f$, C a cube, then C is an implicant of f.
- If $C \subseteq B^n$, and C has k literals, then $|C|$ has 2^{n-k} vertices.

Example 1 $C = x\ y' \subseteq B^3$.

k = 2 , n = 3.

$$C = \{100, 101\}.$$

$|C| = 2 = 2^{3-2}$.

- If k=n, the cube is a minterm.

Representation of Boolean functions

- **The truth table of a function $f : B^n \rightarrow B$ is a tabulation of its value at each of the 2^n vertices of B^n.**
- **For**

$f = \overline{a}\,\overline{b}\,\overline{c}d + \overline{a}\,\overline{b}cd + \overline{a}b\overline{c}d + a\overline{b}\,\overline{c}d + a\overline{b}cd + ab\overline{c}d + abc\overline{d} + abcd$

the truth table is

	abcd	f		abcd	f
0	0000	0	8	1000	0
1	0001	1	9	1001	1
2	0010	0	10	1010	0
3	0011	1	11	1011	1
4	0100	0	12	1100	0
5	0101	1	13	1101	1
6	0110	0	14	1110	1
7	0111	0	15	1111	1

This is intractable for large n (but canonical)

Canonical means that if two functions are the same, then the canonical representations of each are isomorphic.

Sum-of-Products Representation - SOP

- **A function can be represented by a sum of cubes (products):**

f = ab + ac + bc

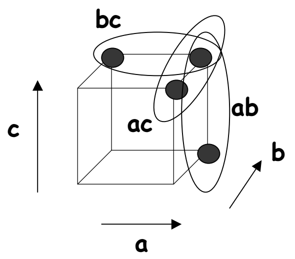
Since each cube is a product of literals, this is a "sum of products" representation

- **A SOP can be thought of as a set of cubes F**

F = {ab, ac, bc} = *C*

- **A set of cubes that represents f is called a cover of f. F={ab, ac, bc} is a cover of f = ab + ac + bc.**

SOP



● = onset minterm
Note that each onset minterm is "covered" by at least one of the cubes, and covers no offset minterm.

- **Covers (SOP's) can efficiently represent many logic functions (i.e. for many, there exist small covers).**
- **Two-level minimization seeks the minimum size cover (least number of cubes)**

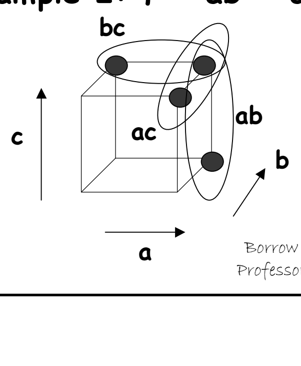
Irredundant

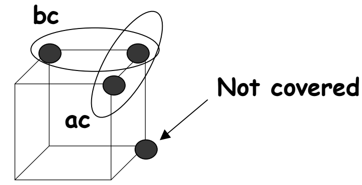
- **Let F = {c_1, c_2, …, c_k} be a cover for f.**

$$f = \sum_{i=1}^{k} c_i$$

A cube $c_i \in$ F is irredundant if $F \setminus \{c_i\} \neq f$

Example 2: f = ab + ac + bc





$F \setminus \{ab\} \neq f$

Prime

- A literal j of cube $c_i \in F$ ($= f$) is prime if

$$(F \setminus \{c_i\}) \cup \{c'_i\} \neq f$$

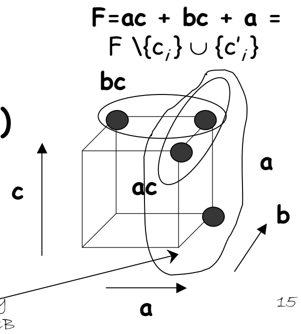
  where c'_i is c_i with literal j of c_i deleted.
- A cube of F is prime if all its literals are prime.

Example 3

f = ab + ac + bc

c_i = ab; c'_i = a (literal b deleted)

F \ $\{c_i\} \cup \{c'_i\}$ = a + ac + bc



Not equal to f since offset vertex is covered

Prime and Irredundant Covers

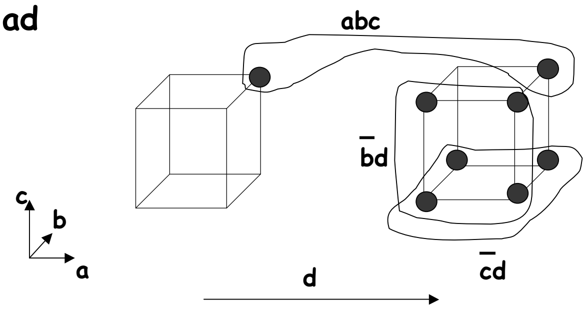
- **Definition 1 A cover is prime (irredundant) if all its cubes are prime (irredundant).**
- **Definition 2 A prime of f is essential (essential prime) if there is a minterm (essential vertex) in that prime but in no other prime.**

Prime and Irredundant Covers

Example 4

$f = abc + \bar{b}d + \bar{c}d$ is prime and irredundant.

abc is essential since $abc\bar{d} \in abc$, but not in $\bar{b}d$ or $\bar{c}d$ or ad



Why is abcd not an essential vertex of abc?

What is an essential vertex of abc?

What other cube is essential? What prime is not essential?

Co-Factor

Shannon (Boole) cofactors

Let $f : B^n \rightarrow B$ be a Boolean function, and $x = (x_1, x_2, \ldots, x_n)$ the variables in the support of f. The cofactor f_a of f by a literal $a=x_i$ or $a=\overline{x}_i$ is

$$f_{x_i}(x_1, x_2, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)$$
$$f_{\overline{x}_i}(x_1, x_2, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$$

Shannon (Boole) Cofactor

- **The cofactor f_C of f by a cube C is just f with the fixed values indicated by the literals of C, e.g. if $C=x_i \overline{x}_j$, then $x_i =1$, and $x_j =0$.**
- **If $C= x_1 \overline{x}_4 x_6$, f_C is just the function f restricted to the subspace where $x_1 =x_6 =1$ and $x_4 =0$.**
- **As a function, f_C does not depend on x_1, x_4 or x_6 (However, we still consider f_C as a function of all *n* variables, it just happens to be independent of x_1, x_4 and x_6).**
- **$x_1 f \neq f_{x_1}$**

Example: $f= ac + \overline{a}\,\overline{c}$, $af = ac$, $f_a=c$

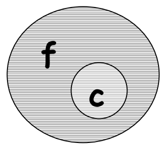
Fundamental Theorem

Theorem 1 Let c be a cube and f a function. Then $c \subseteq f \Leftrightarrow f_c \equiv 1$.

Proof. We use the fact that $xf_x = xf$, and f_x is independent of x.

If: Suppose $f_c \equiv 1$. Then $cf = f_c c = c$. Thus,

$c \subseteq f$.



Proof (contd)

Only if. Assume $c \subseteq f$

Then $c \subseteq cf = cf_c$. But f_c is independent of literals $l \in c$. If $f_c \neq 1$, then $\exists\, m \in B^n$, $f_c(m)=0$.

Let $m_i' = m_i$, if $x_i \notin c$ and $\overline{x}_i \notin c$.

or if $m_i=0$, $\overline{x}_i \in c$

or $m_i=1$, $x_i \in c$.

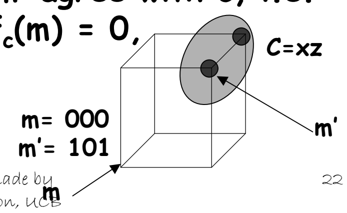
$m_i' = \overline{m}_i$ otherwise.

i.e. we make the literals of m' agree with c, i.e. $m' \in c$. But then $f_c(m') = f_c(m) = 0$,

Hence, $c(m')=1$

and $f_c(m')\, c(m') = 0$,

contradicting $c \subseteq cf_c$.



Cofactor of Covers

Definition 3 The cofactor of a cover F is the sum of the cofactors of each of the cubes of F.

Note: If $F=\{c_1, c_2, \ldots, c_k\}$ is a cover of f, then $F_c=\{(c_1)_c, (c_2)_c, \ldots, (c_k)_c\}$ is a cover of f_c.

Suppose F(x) is a cover of f(x), i.e.

$$F(x) = \sum_i c_i = \sum_i \prod_j \ell^i_j = \{c_i\}$$

Then for $1 \le j \le n$,

$$F(x)_{x_j} = \sum_i (c_i)_{x_j}$$

is a cover of $f_{x_j}(x)$

Definition 4 **The cofactor C_{x_j} of a cube C with respect to a literal x_j is**

- **C if x_j and $\overline{x}_j$ do not appear in C**
- **$C\backslash\{x_j\}$ if x_j appears positively in C, i.e. $x_j \in C$**
- **$\emptyset$ if x_j appears negatively in C, i.e. $\overline{x}_j \in C$**

Example 5

If $C = x_1 \overline{x}_4 x_6$,

$C_{x_2} = C$ (x_2 and $\overline{x}_2$ do not appear in C)

$C_{x_1} = \overline{x}_4 x_6$ (x_1 appears positively in C)

$C_{x_4} = \emptyset$ (x_4 appears negatively in C)

Example 6

$F = abc + \overline{b}d + \overline{c}d$

$F_b = ac + \overline{c}d$

(Just drop b everywhere and throw away cubes containing literal $\overline{b}$)

Borrow from slides made by
Professor R. K. Brayton, UCB

25

Shannon Expansion

f : $B^n \rightarrow B$

Theorem 2

$$f = x_i f_{x_i} + \overline{x}_i f_{\overline{x}_i}$$

Theorem 3 F is a cover of f. Then

$$\tilde{F} \equiv x_i F_{x_i} + \overline{x}_i F_{\overline{x}_i}$$

We say that f (F) is expanded about x_i. x_i is called the splitting variable.

Borrow from slides made by
Professor R. K. Brayton, UCB

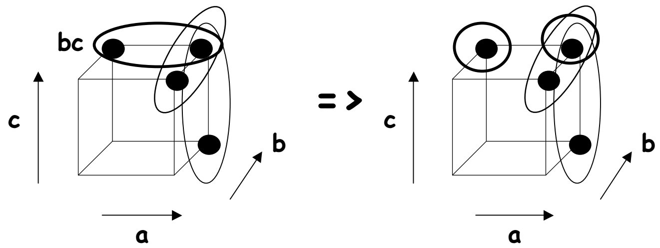
26

Example 7

$$F = ab + ac + bc$$

$$\tilde{F} = aF_a + \overline{a}F_{\overline{a}} = a(b + c + bc) + \overline{a}(bc)$$

$$= ab + ac + abc + \overline{a}bc$$



Cube bc got split into two cubes

Cover Matrix

We sometimes use matrix notation to represent a cover:

Example 8 F = ac + $\overline{c}$d =

	a	b	c	d		a	b	c	d
ac→	1	2	1	2	or	1	-	1	-
$\overline{c}$d→	2	2	0	1		-	-	0	1

Each row represents a cube. 1 means that the positive literal appears in the cube, 0 the negative. The 2 (or -) here represents that the variable does not appear in the cube. It also represents both 0 and 1 values.

Example 9

$a\bar{c}$ = 1 2 0 2 = {1, {0,1}, 0, {0,1}}
= {1000, 1100, 1001, 1101}

2 is sometimes called "input don't care", but this is confusing so we won't use the term.

Incompletely Specified Functions

F = (f, d, r) : $B^n \rightarrow$ {0, 1, *}
where * represents "don't care". (Sometimes we use 2 in place of *)

- f = onset function - $f(x)=1 \leftrightarrow F(x)=1$
- r = offset function - $r(x)=1 \leftrightarrow F(x)=0$
- d = don't care function - $d(x)=1 \leftrightarrow F(x)=*$

(f,d,r) forms a partition of B^n. i.e.
- $f + d + r = B^n$
- $fd = fr = dr = \emptyset$ (pairwise disjoint)

A completely specified function *g* is a cover for *F*=(f,d,r) if

$$f \subseteq g \subseteq f+d$$

(Thus *gr* = ∅). Thus, if x∈d (i.e. d(x)=1), then g(x) can be 0 or 1, but if x∈f, then g(x)=1 and if x∈r, then g(x)=0.

(We "don't care" which value *g* has at x∈d)

Primes of Incompletely Specified Functions

Definition 5 A cube c is prime of *F*=(f,d,r) if c⊆f+d (an implicant of f+d), and no other implicant (of f+d) contains c, i.e.

$$\forall \tilde{c}, \tilde{c} \subseteq f+d, c \not\subset \tilde{c}$$

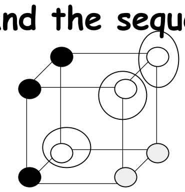
(i.e. it is simply a prime of f+d)

Definition 6 Cube c_j of cover F={c_i} is redundant if f⊆ F\{c_j}. Otherwise it is irredundant.

Note that c⊆f+d ↔ cr = ∅

Example:Logic Minimization

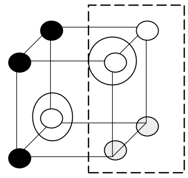
Consider $F(a,b,c)=(f,d,r)$, where $f=\{\overline{a}b\overline{c}, a\overline{b}c, abc\}$ and $d=\{a\overline{b}\overline{c}, ab\overline{c}\}$, and the sequence of covers illustrated below:



$F^1 = \overline{a}b\overline{c} + a\overline{b}c + abc$

Expand $abc \rightarrow a$

$F^2 = a + \overline{a}b\overline{c} + a\overline{b}c$



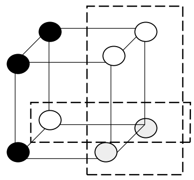
$a\overline{b}c$ is redundant

a is prime

$F^3 = a + \overline{a}b\overline{c}$

Expand $\overline{a}b\overline{c} \rightarrow b\overline{c}$

$F^4 = a + b\overline{c}$



○ **on**

● **off**

◌ **Don't care**

Borrow from slides made by Professor R. K. Brayton, UCB

33