Synthesis of 2-level Logic

Exact and Heuristic Methods

Lecture 7: Branch & Bound

Two Approaches

- Exact
  - Find all primes
    - Find a complete sum
  - Find a minimum cover (covering problem)
- Heuristic
  - Take an initial cover of cubes
  - Repeat
    - Expand a cube
    - Remove another cube
  - Eliminate consensus terms

From Lecture 6

- We have learned how to compute primes
- We have learned how to build constraint matrix
- Now we are going to see how to solve the minimum covering problem

Example

- Primes: wy'z, wyz', wxy, wxz, x'y', x'z'
- f = x'y' + wxy + x'yz' + wy'z

	wy'z	wyz'	wxy	wxz	x'y'	x'z'
x'y'	0	0	0	0	1	0
wxy	0	0	1	0	0	0
x'yz'	0	0	0	0	0	1
wy'z	1	0	0	0	0	0

constraint matrix

F = x'z' + x'y' + wxy + wy'z

Problem Re-Formulation

	x'y	x'z'	y'z'	yz
	p1	p2	p3	p4
x'y'z'	0	1	1	0
x'yz'	1	1	0	0
x'yz	1	0	0	1
xyz	0	0	0	1
xy'z'	0	0	1	0

(p2+p3)(p1+p2)(p1+p4)p3p4=1

- Problem: Given a boolean formula f, find a minimum assignment to satisfy f (to make f=1)
  - A minimum assignment is the one with the minimum cost
  - The cost of a "p" can be measured by the required gates to implement the prime

Unate Covering Problem

- Given n variables P={p1,p2,...,pn} and a POS formula F=(...)(...)(...)...(...), find the minimum subset of P
  - S.t. assigning 1 to all variables in P makes F=1
- It is called "unate" because all variables in F are unate variables

Definition 1 A function $f : B^n \rightarrow B$ is **positive unate in variable** x_i **iff**

$$f_{\bar{x}_i} \subseteq f_{x_i}$$

This is equivalent to monotone increasing in x_i**:**

$$f(m^-) \leq f(m^+)$$

Similarly for negative unate monotone decreasing:

$$f_{x_i} \subseteq f_{\bar{x}_i}$$

$$f(m^-) \geq f(m^+)$$

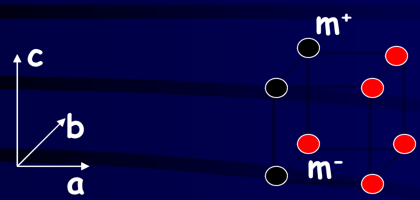
A function is unate in x_i if it is either positive unate or negative unate in x_i.

Definition 2 A function is unate if it is unate in each variable.

Definition 3 A cover f is positive unate in x_i iff $\bar{x}_i \notin c_j$ for all cubes (terms) $c_j \in F$

Example 1

$f = ab + b\bar{c} + a\bar{c}$



positive unate in a,b
negative unate in c

$f(m^-)=1 \geq f(m^+)=0$

1st step: Reduction of Matrix

- Eliminate rows by essential columns
- Eliminate rows by row dominance
- Eliminate columns by column dominance

Essential column

- p3 and p4 are essential
  - So the two rows can be eliminated

	x’y	x’z’	y’z’	yz
	p1	p2	p3	p4
x’y’z’	0	1	1	0
x’yz’	1	1	0	0
x’yz	1	0	0	1
xyz	0	0	0	1
x’y’z’	0	0	1	0

essential

Row dominance

- If row i dominate row j, then remove row i
  - Row i can be covered for sure

$$\begin{array}{c} \\ 1\\2\\3\\4\\5\\6 \end{array}\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccccc} 1 & 1 & & & & 1 \\ 1 & 1 & & & & \\ & 1 & 1 & & & \\ & 1 & 1 & 1 & & \\ & & & 1 & 1 & \\ & & & 1 & 1 & 1 \end{array}\right] \end{array} \Rightarrow \begin{array}{c} \\ 2\\3\\5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{ccccc} 1 & 1 & & & \\ & 1 & 1 & & \\ & & & 1 & 1 \end{array}\right] \end{array}$$

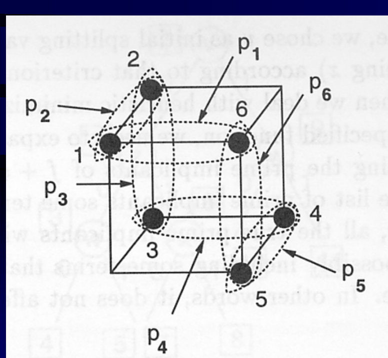
Column dominance

- If a prime *pi* has an equal or lower cost than a prime *pj*,
- And, *pi* is column-dominance of *pj*
  - Then, we don't want to select *pj*
  - Because *pi* covers more min-terms
- The "cost" of a prime is measured by the AND gate implementing it
  - Ex. xyz has a higher cost than xz

$$
\begin{array}{c}
 \\ 2 \\ 3 \\ 5
\end{array}
\begin{array}{c}
\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{bmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & & 1 & 1 \end{bmatrix}
\end{array}
\quad \Rightarrow \quad
\begin{array}{c}
 \\ 2 \\ 3 \\ 5
\end{array}
\begin{array}{c}
\begin{array}{cc} 2 & 4 \end{array} \\
\begin{bmatrix} 1 & \\ 1 & \\ & 1 \end{bmatrix}
\end{array}
$$

Cyclic core

- For this type of matrix, we cannot say for sure which primes should stay
- We need to search for the lowest-cost answer



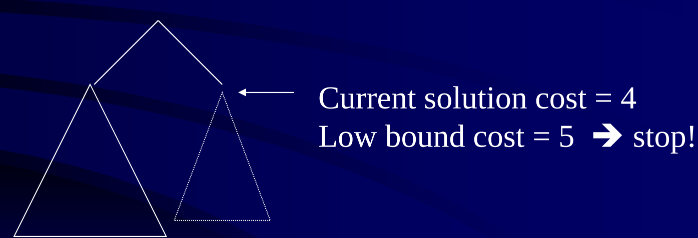
$$
\begin{array}{c|cccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & 1 & & & & & 1 \\
2 & 1 & 1 & & & & \\
3 & & 1 & 1 & & & \\
4 & & & 1 & 1 & & \\
5 & & & & 1 & 1 & \\
6 & & & & & 1 & 1
\end{array}
$$

Systematic search

- how do we do a systematic search?
  - Pick a variable
  - Split it into two cases
    - Set the variable 1
    - Set the variable 0
  - Try to stop as early as possible
    - Without exploring the entire search sub-tree
- This is a typical paradigm called "branch and bound"
  - See your algorithm textbook

How to bound?

- By quickly computing the lower bound of the cost associated with a sub-tree
- If that lower bound is > the current best solution, then there is no need to proceed



Lower bound = MIS

- Finding the lower bound number is to identify the maximal independent set
  - We want this number as big as possible

$$
\begin{array}{c|c}
 & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\
\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} &
\begin{bmatrix}
1 & & & & & 1 \\
1 & 1 & & & & \\
 & 1 & 1 & & & \\
 & & 1 & 1 & & \\
 & & & 1 & 1 & \\
 & & & & 1 & 1
\end{bmatrix}
\end{array}
$$

- For example, the 1st, 3rd, and 5th rows are independent
- Hence, we need at least 3 columns (primes) to cover the rows
- This represents the lowest (possible) number of primes required

The obvious lower bound

- The obvious lower bound is 2
  - Theorem 4.8.1 (orange textbook)
- Otherwise, there is a column (prime) dominating everything else

Algorithm to compute MIS

```
MIS_QUICK(M) {
    MIS = ∅
    do {
        i = CHOOSE_SHORTEST_ROW(M)
        MIS = MIS ∪ {i}
        M = DELETE_INTERSECTING ROWS(M, i)
    } while (|| M ||> 0) continue
    return (MIS)
}
```

- Define the "shortest"
  - 1. Count the "1" in a row
  - 2. Count the "1" by including all "1" vertically (columns)

MIS

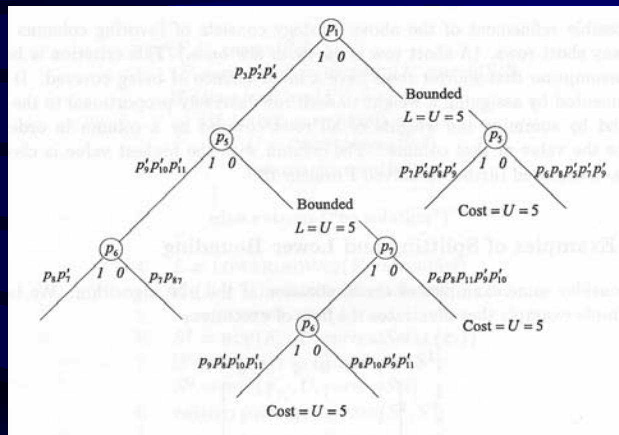
$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{array}{lll} 1 & w_1^1 = 2 & w_1^2 = 5 \\ 2 & w_2^1 = 3 & w_2^2 = 9 \\ 3 & w_3^1 = 2 & w_3^2 = 6 \\ 4 & w_4^1 = 3 & w_4^2 = 7 \\ 5 & w_5^1 = 2 & w_5^2 = 6 \\ 6 & w_6^1 = 2 & w_6^2 = 5 \\ 7 & w_7^1 = 2 & w_7^2 = 6 \end{array}$$

Heuristic 2.

$$M = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{array}{lll} 3 & w_3^1 = 2 & w_3^2 = 4 \\ 5 & w_5^1 = 2 & w_5^2 = 3 \\ 6 & w_6^1 = 2 & w_6^2 = 3 \end{array}$$

- Heuristic 1:
  - Select {1}, Select {3}, output {1,3} = lower bound 2
- Heuristic 2:
  - Select {1}, select {5}, select {6}, output {1,5,6} = lower bound 3

The Branch and Bound



- Use lower bound to prune search space

The Branch and Bound Algorithm

```
BCP(F, U, currentSol) {
    (F, currentSol) = REDUCE(F, currentSol)
    if (terminalCase(F)) {
        if (COST(currentSol) < U) {
            U = COST(currentSol)
            return (currentSol)
        }
        else return ("no solution")
    }
    L = LOWER_BOUND(F, currentSol)
    if (L ≥ U) return ("no solution")
    x_i = CHOOSE_VAR(F)
    S^1 = BCP(F_{x_i}, U, currentSol ∪ {x_i})
    if (COST(S^1) = L) return (S^1)
    S^0 = BCP(F_{x_i'}, U, currentSol)
    return BEST_SOLUTION(S^1, S^0)
}
```

Example 1

- [set upper bound = 7] [MIS set lower bound = 2]
  - Upper bound is the max size of current solution
- [split on prime 1] [get a solution {1,2} = lower bound]
- [stop]

$$
\begin{array}{c}
\\ 1 \\ 2 \\ 3 \\ 4
\end{array}
\begin{array}{c}
\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\left[\begin{array}{cccccc}
1 & & 1 & & 1 & \\
1 & & & 1 & & 1 \\
& 1 & 1 & & & 1 \\
& 1 & & 1 & 1 &
\end{array}\right]
\end{array}
$$

Example 2

- [upper = 7] [lower = 3]
- [split on column 1]
- [p2 and p6 are dominated] [remove p2 and p6]
- [p3 and p5 become essential]
- [answer {1,3,5}]

$$
\begin{array}{c}
\\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6
\end{array}
\begin{array}{c}
\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\left[\begin{array}{cccccc}
1 & & & & & 1 \\
1 & 1 & & & & \\
& 1 & 1 & & & \\
& & 1 & 1 & & \\
& & & 1 & 1 & \\
& & & & 1 & 1
\end{array}\right]
\end{array}
\qquad
\begin{array}{c}
\\ 3 \\ 4 \\ 5 \\ 6
\end{array}
\begin{array}{c}
\begin{array}{ccccc} 2 & 3 & 4 & 5 & 6 \end{array} \\
\left[\begin{array}{ccccc}
1 & 1 & & & \\
& 1 & 1 & & \\
& & 1 & 1 & \\
& & & 1 & 1
\end{array}\right],
\end{array}
$$

Example 3

- [upper = 12] [MIS gives lower bound = 4]

$$
M = \begin{array}{c}
\begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \end{array} \\
\left[\begin{array}{cccc|ccccccc}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0
\end{array}\right]
\end{array}
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \end{array}
$$

Set p1 = 1

- Rows 1,4,12 are covered
- Columns 2 and 4 are dominated
- Column 3 becomes essential

$$
M = \begin{array}{c}
\begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \end{array} \\
\left[\begin{array}{cccc|ccccccc}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0
\end{array}\right]
\end{array}
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \end{array}
$$

p1=1

$$
M_{p_1} = \begin{array}{c}
\begin{array}{ccccccc} 5 & 6 & 7 & 8 & 9 & 10 & 11 \end{array} \\
\left[\begin{array}{ccccccc}
1 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 \\
 & & & & & & \\
1 & 1 & 0 & 1 & 0 & 0 & 1
\end{array}\right]
\end{array}
\begin{array}{c} \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ \\ 13 \end{array}
$$

Set p5=1

- Current solution is {p1,p3,p5}

$$M_{p_1} = \begin{array}{c} \begin{array}{ccccccc} 5 & 6 & 7 & 8 & 9 & 10 & 11 \end{array} \\ \left[\begin{array}{ccccccc} 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \end{array} \begin{array}{c} \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 13 \end{array}$$

p5=1

$$M_{p_1 p_5} = \begin{array}{c} \begin{array}{ccc} 6 & 7 & 8 \end{array} \\ \left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{array}\right] \end{array} \begin{array}{c} \\ 6 \\ 7 \\ 8 \end{array}$$

This is a cyclic core

- [set p6=1] [choose either p7 or p8]
- Answer {p1, p3, p5, p6, p7}
- Upper bound now is 5

$$M_{p_1 p_5} = \begin{array}{c} \begin{array}{ccc} 6 & 7 & 8 \end{array} \\ \left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{array}\right] \end{array} \begin{array}{c} \\ 6 \\ 7 \\ 8 \end{array}$$

Backtrack to "p5=0"

- The lower bound = 5. There is no need to proceed to the sub-tree

$$M_{p_1} = \begin{array}{c} \begin{array}{ccccccc} 5 & 6 & 7 & 8 & 9 & 10 & 11 \end{array} \\ \left[\begin{array}{ccccccc} 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \end{array} \begin{array}{c} \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 13 \end{array}$$

p5=0 (remove column 5)

$$M_{p_1 p_3, p_5'} = \begin{array}{c} \begin{array}{cccccc} 6 & 7 & 8 & 9 & 10 & 11 \end{array} \\ \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ 5 \\ 6 \\ 7 \\ 9 \\ 10 \\ 11 \\ 13 \end{array} \quad \Rightarrow L = 2 + 3 = 5$$

Backtrack to "p1=0"

- p2, p4, p11 are essential. Remove them
- P10, p13 are dominated by p5

$$M = \begin{array}{c} \begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \end{array} \\ \left[\begin{array}{ccccccccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \end{array}$$

p1=0

$$M_{p_1'} = \begin{array}{c} \begin{array}{cccccccccc} 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \end{array} \\ \left[\begin{array}{cccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \end{array}$$

Lower bound check

- Current partial solution = {p2, p4, p11}
- The reduced matrix has a lower bound 2
- The total cost lower bound is again = 5
- Stop.

$$
M_{p_1'} = \begin{array}{c}
\begin{array}{ccccccccccc} 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \end{array} \\
\left[\begin{array}{ccccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0
\end{array}\right]
\end{array}
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \end{array}
$$

$$
\begin{array}{c}
\begin{array}{ccccc} 5 & 6 & 7 & 8 & 9 \end{array} \\
\left[\begin{array}{ccccc}
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 1
\end{array}\right]
\end{array}
\begin{array}{c} \\ 5 \\ 6 \\ 7 \\ 10 \end{array}
$$